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Division algorithm :-

Let $R[x]$ be the ring of polynomials over a ring R and let $f(x), g(x)$ in $R[x]$ be two polynomials. Then there exists unique polynomials $q(x), r(x)$ in $R[x]$ such that

$$f(x) = q(x)\,g(x) + r(x)$$

& $0 \leq \deg[r(x)] < \deg[g(x)]$

(1)

Proof :- Consider set of polynomials

$$S(x) = \{ f(x) - t(x) g x \,/\, \deg(f(x) - t(x) g(x)) \geq 0 \}$$

Case (i) :- If $0 \in S(x)$,

then there exists $t(x)$ such that

$$f(x) - t(x) g(x) = 0$$

ie, $f(x) = t(x) g(x)$

Now the result is true with

quotient $q(x) = t(x)$,

remainder $r(x) = 0$.

Case (ii) If $0 \notin S(x)$ (2)

Claim: $0 \leq \deg(r(x)) < \deg g(x)$

Suppose not,

ie, $r(x) = f(x) - q(x)g(x)$

but $\deg(r(x)) \geq \deg g(x)$

Take $r(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$

$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$

$r(x) = a_n x^n + a_{n-1} x^{n-1} + \quad + a_0$

$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0$

with $\deg(r(x)) > \deg g(x)$

ie, $n > m$

$$= \quad r(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

$$x^{n-m} a_n b_m^{-1} g(x) = a_n b_m b_m^{-1} x^n \quad — \quad (1)$$

$$a_n b_m^{-1} b_{m-1} x^{n-1} + \cdots + b_0 x^{n-m}$$

$$\Rightarrow r(x) - a_n b_m^{-1} x^{n-m} g(x) =$$

$$(a_{n-1} - a_n b_m^{-1} b_{m-1}) x^{n-1} + \cdots + a_0 \longrightarrow (2)$$

Take it as $h(x)$

ie, $h(x) = r(x) - a_n b_n^{-1} x^{n-m} g(x)$

Now the form of $h(x)$ is

$h(x) = f(x) - q(x) g(x) -$

$a_n b_m^{-1} x^{n-m} g(x)$

$= f(x) - [q(x) + a_n b_m^{-1} x^{n-m}] g(x)$

$= f(x) - t(x) g(x)$

ie, $h(x) \in S(x)$

but $\deg(h(x)) = n-1 <$

$\deg(r(x))$

$\Rightarrow h(x)$ is smaller than $r(x)$ in $S(x)$

which is a contradiction

$\therefore \; 0 \leq \deg(r(x)) < \deg(g(x))$

Uniqueness:-

If there are $q_1(x), r_1(x)$, $q_2(x), r_2(x)$ such that

$f(x) = q_1(x)\, g(x) + r_1(x)$

$f(x) = q_2(x)\, g(x) + r_2(x)$

Subtracting,

$$(q_1(x) - q_2(x))g(x) + (r_1(x) - r_2(x)) = 0$$

$$\Rightarrow [q_1(x) - q_2(x)]\,g(x) = r_2 x - r_1(x)$$

If $q_1(x) - q_2(x) \neq 0$

Then $\deg(r_2(x) - r_1(x)) = \deg[q_1(x) - q_2(x)]\,g(x)$

$> \deg(g(x)) \rightarrow$ which is not possible

$\therefore q_1(x) - q_2(x) = 0$

hence $q_1(x) = q_2(x)$ & $r_2(x) - r_1(x) = 0$

(7)

$$\Rightarrow q_1(x) = q_2(x) \text{ \& }$$

$$r_1(x) = r_2(x)$$

$\therefore$ Uniqueness proved...

Hence division algorithm

proved.